

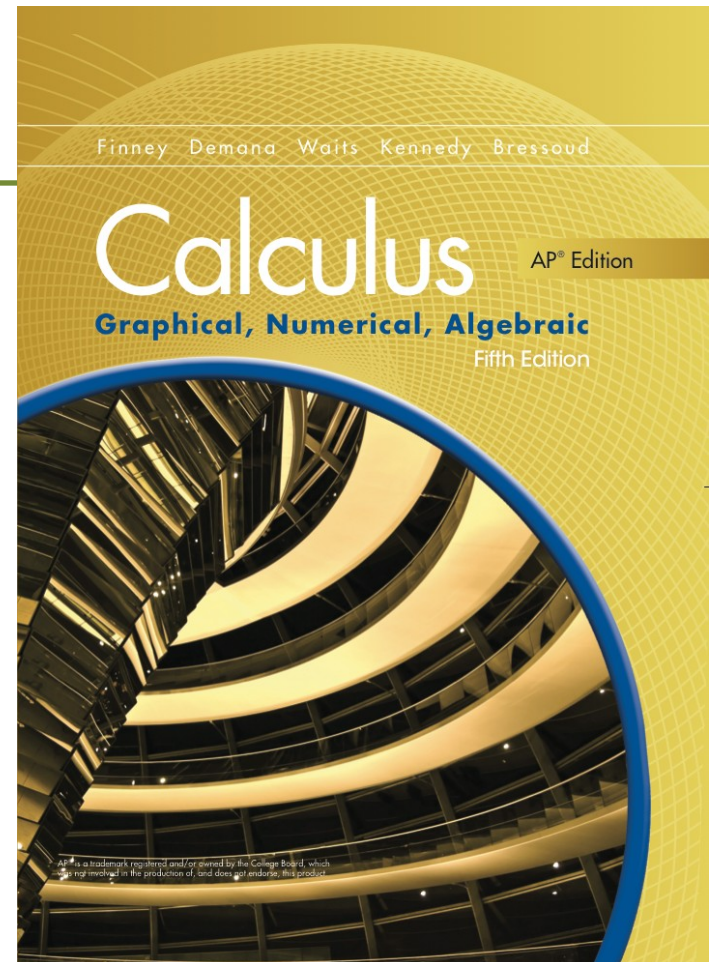
Chapter 11

Parametric, Vector, and Polar Functions



Section 11.2

Vectors in the Plane



Quick Review

Let $P = (1, 3)$ and $Q = (4, 7)$.

1. Find the distance between the points P and Q .
2. Find the slope of the line segment PQ .
3. If $R = (3, b)$, determine b so that segments PQ and RQ are collinear.
4. If $R = (3, b)$, determine b so that segments PQ and RQ are perpendicular.
5. Determine the missing coordinate so that the four points form a parallelogram $ABCD$. $A = (1, 1)$, $B = (3, 5)$, $C = (8, b)$, $D = (6, 2)$

Quick Review

6. Find the velocity and acceleration of a particle moving along a line if its position at time t is given by $s(t) = t \cos t$
7. A particle moves along the x -axis with velocity $v(t) = 2t + 6$ for $t \geq 0$. If its position is $x = 10$ when $t = 0$, where is the particle when $t = 6$?
8. Find the length of the curve defined parametrically by $x = \sin 2t$ and $y = \cos 3t$ for $0 \leq t \leq 2\pi$.

Quick Review Solutions

Let $P = (1, 3)$ and $Q = (4, 7)$.

1. Find the distance between the points P and Q . 5

2. Find the slope of the line segment PQ . $\frac{4}{3}$

3. If $R = (3, b)$, determine b so that segments PQ and RQ are collinear.

$b = \frac{17}{3}$

4. If $R = (3, b)$, determine b so that segments PQ and RQ are perpendicular.

$b = \frac{31}{4}$

5. Determine the missing coordinate so that the four points form a parallelogram $ABCD$. $A = (1, 1)$, $B = (3, 5)$, $C = (8, b)$, $D = (6, 2)$

$b = 6$

Quick Review Solutions

6. Find the velocity and acceleration of a particle moving along a line if its position at time t is given by $s(t) = t \cos t$

$$v(t) = -t \sin t + \cos t \quad a(t) = -t \cos t - 2 \sin t$$

7. A particle moves along the x -axis with velocity $v(t) = 2t + 6$ for $t \geq 0$. If its position is $x = 10$ when $t = 0$, where is the particle when $t = 6$?

$$s(6) = 82$$

8. Find the length of the curve defined parametrically by $x = \sin 2t$ and $y = \cos 3t$ for $0 \leq t \leq 2\pi$. $L = 15.28937$

What you'll learn about

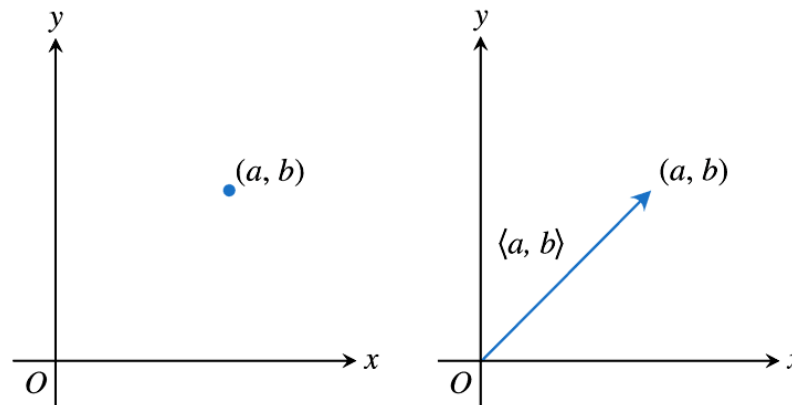
- Vector operations
- Velocity and acceleration for vector-valued functions
- Displacement and distance traveled for vector-valued functions

...and why

The jump from one to two dimensions (and eventually higher) is easier than one might think, thanks to the mathematics of vectors.

Two-Dimensional Vectors

A **two-dimensional vector** $\mathbf{v}$ is an ordered pair of real numbers, denoted in **component form** as $\langle a, b \rangle$. The numbers a and b are the **components** of the vector $\mathbf{v}$. The **standard representation** of the vector $\langle a, b \rangle$ is the arrow from the origin to the point $\langle a, b \rangle$. The **magnitude** of $\mathbf{v}$, denoted $|\mathbf{v}|$, is the length of the arrow, and the **direction of** $\mathbf{v}$ is the direction in which the arrow is pointing. The vector $\mathbf{0} = \langle 0, 0 \rangle$, called the **zero vector**, has zero length and no direction.



Magnitude of a Vector

The **magnitude** or **absolute value** of the vector $\langle a, b \rangle$ is the nonnegative real number $|\langle a, b \rangle| = \sqrt{a^2 + b^2}$.

Direction Angle of a Vector

The **direction angle** of a nonzero vector $\mathbf{v}$ is the smallest nonnegative angle θ formed with the positive x -axis as the initial ray and the standard representation of $\mathbf{v}$ as the terminal ray.

Head Minus Tail (HMT) Rule

If an arrow has initial point (x_1, y_1) and terminal point (x_2, y_2) , it represents the vector $\langle x_2 - x_1, y_2 - y_1 \rangle$.

Example Finding Magnitude and Direction

Find the magnitude and the direction angle θ if the vector $\mathbf{v} = \langle 1, \sqrt{3} \rangle$.

The magnitude of $\mathbf{v}$ is $|\mathbf{v}| = \sqrt{1^2 + \sqrt{3}^2} = 2$.

The direction angle θ must satisfy $\cos\theta = 1/2$ and $\sin\theta = \sqrt{3}/2$.
 $\theta = \pi/3$ or 60° .

Vector Addition and Scalar Multiplication

Let $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$ be vectors and let k be a real number (scalar).

The **sum (or resultant) of the vectors $\mathbf{u}$ and $\mathbf{v}$** is the vector

$$\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2 \rangle.$$

The **product of the scalar k and the vector $\mathbf{u}$** is

$$k\mathbf{u} = k\langle u_1, u_2 \rangle = \langle ku_1, ku_2 \rangle.$$

The **opposite of a vector $\mathbf{v}$** is $-\mathbf{v} = (-1)\mathbf{v}$. We define vector subtraction by

$$\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v}).$$

Example Performing Operations on Vectors

Let $\mathbf{u} = \langle -1, 2 \rangle$ and $\mathbf{v} = \langle 4, 8 \rangle$. Find the following.

(a) $2\mathbf{u} + 3\mathbf{v}$

(b) $\mathbf{u} - \mathbf{v}$

(c) $|1/2\mathbf{v}|$

(a) $2\mathbf{u} + 3\mathbf{v} = 2\langle -1, 2 \rangle + 3\langle 4, 8 \rangle = \langle -2, 4 \rangle + \langle 12, 24 \rangle = \langle 10, 28 \rangle$

(b) $\mathbf{u} - \mathbf{v} = \langle -1, 2 \rangle + \langle -4, -8 \rangle = \langle -5, -6 \rangle$

(c) $1/2\mathbf{v} = \langle 2, 4 \rangle$

$|1/2\mathbf{v}| = \sqrt{4 + 16} = 2\sqrt{5}$

Properties of Vector Operations

Let $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ be vectors and a , b be scalars.

1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$

3. $\mathbf{u} + \mathbf{0} = \mathbf{u}$

4. $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$

5. $0\mathbf{u} = \mathbf{0}$

6. $1\mathbf{u} = \mathbf{u}$

7. $a(b\mathbf{u}) = (ab)\mathbf{u}$

8. $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$

9. $(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$

Example Finding Ground Speed and Direction

A plane flying due east at 400 mph in still air, encounters a 50-mph tail wind acting in the direction 60° north of east. The plane holds its compass heading due east but, because of the wind, acquires a new ground speed and direction. What are they?

Let $\mathbf{u}$ =velocity of plane in still air and $\mathbf{v}$ =velocity of the tail wind, then $|\mathbf{u}| = 400$ and $|\mathbf{v}| = 50$. It follows that $\mathbf{u} = \langle 400, 0 \rangle$ and $\mathbf{v} = \langle 50 \cos 60^\circ, 50 \sin 60^\circ \rangle$ and $\mathbf{u} + \mathbf{v} = \langle 400 + 25, 0 + 25\sqrt{3} \rangle$.

The ground speed is equal to $|\mathbf{u} + \mathbf{v}| = \sqrt{425^2 + (25\sqrt{3})^2} = 427.2$ mph

The direction is equal to $\theta = \tan^{-1} \frac{25\sqrt{3}}{425} = 5.8^\circ$ north of east.

Velocity, Speed, Acceleration, and Direction of Motion

Suppose a particle moves along a smooth curve in the plane so that its position at any time t is $(x(t), y(t))$, where x and y are differentiable functions of t .

1. The particle's **position vector** is $\mathbf{r}(t) = \langle x(t), y(t) \rangle$.
2. The particle's **velocity vector** is $\mathbf{v}(t) = \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle$.
3. The particle's **speed** is the magnitude of $\mathbf{v}$, denoted $|\mathbf{v}|$. Speed is a scalar.
4. The particle's **acceleration vector** is $\mathbf{a}(t) = \left\langle \frac{d^2 x}{dt^2}, \frac{d^2 y}{dt^2} \right\rangle$.
5. The particle's **direction of motion** is the **direction vector** $\frac{\mathbf{v}}{|\mathbf{v}|}$.

Displacement and Distance Traveled

Suppose a particle moves along a path in the plane so that its velocity at any time t is $\mathbf{v}(t) = (v_1(t), v_2(t))$, where v_1 and v_2 are integrable functions of t .

The **displacement** from $t = a$ to $t = b$ is given by the vector

$$\left\langle \int_a^b v_1(t) dt, \int_a^b v_2(t) dt \right\rangle.$$

The preceding vector is added to the position at $t = a$ to get the position at $t = b$.

The **distance traveled** from $t = a$ to $t = b$ is

$$\int_a^b |\mathbf{v}(t)| dt = \int_a^b \sqrt{(v_1(t))^2 + (v_2(t))^2} dt.$$

Example Finding Displacement and Distance Traveled

A particle moves in the plane with velocity vector

$$\mathbf{v}(t) = (t - 3\pi \cos \pi t, 2t - \pi \sin \pi t).$$

At $t = 0$, the particle is at the point $(1, 5)$.

(a) Find the position of the particle at $t = 4$.

(b) What is the total distance traveled by the particle from $t = 0$ to $t = 4$?

$$(a) \text{ Displacement} = \left\langle \int_0^4 (t - 3\pi \cos \pi t) dt, \int_0^4 (2t - \pi \sin \pi t) dt \right\rangle = \langle 8, 16 \rangle.$$

The particle is at point $\langle 1 + 8, 5 + 16 \rangle = \langle 9, 21 \rangle$

$$(b) \text{ Distance traveled} = \int_0^4 \sqrt{(t - 3\pi \cos \pi t)^2 + (2t - \pi \sin \pi t)^2} dt = 33.533.$$